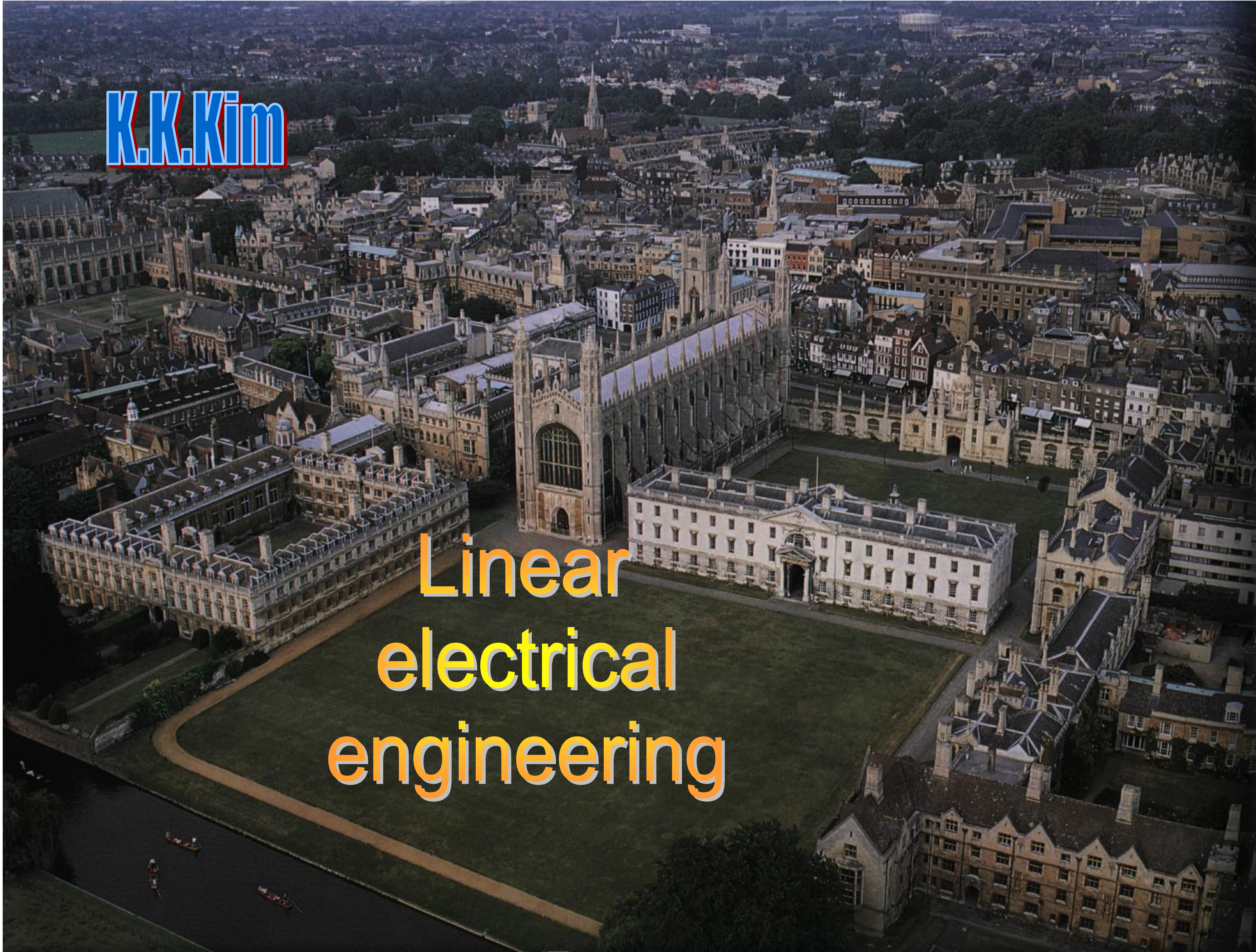

K.K.Kim

Linear electrical engineering

1. FUNDAMENTALS OF THEORY OF LINEAR ELECTRICAL CIRCUITS

1.1. The electrical circuit and its elements

The electrical circuit is a combination of devices intended for passing an electric current in them. Electromagnetic processes can be described with such terms as a current, a voltage, (potential difference), an electromotive force, a charge, a magnetic flow, a resistance, an inductance, a mutual inductance and a capacitance in it.

The main elements of a circuit are *sources* and *loads* of electrical energy (signals).

The *power sources* (signals) are intended for converting various kinds of energy into electrical energy (turbo and hydrogenerators, accumulators, electron generators etc.)

The *loads of energy* (signals) are used for converting electrical energy into other kinds of energy (electrical motors, electrical furnaces, electron-beam tubes etc.)

Besides these main elements, the circuit contains various auxiliary elements, which link the sources with the loads (lines of transmissions, connective wires), suppress or strengthen a certain component of the signal (filters, amplifiers), change the value of voltages and currents (transformers) and so on.

There are two kinds of circuits: the first circuits are intended for transmitting and converting electrical energy (these circuits are used in electro-power engineering) and the second circuits for transmitting and transforming the information (these circuits are used in communication engineering, radio engineering, devices of automation and telemechanics etc.)

Each element of the circuit has a certain number of terminals (poles), with their help this element is connected with other elements. *Two-pole elements* (two-terminal networks) have two terminals. Power sources (except multiphase and controllable sources), resistors, capacitors, inductance coils are two-pole elements. Three-pole elements are electron lamps (vacuum triodes) and transistors (semiconductive triodes). Examples of four-pole elements are two-winding transformers, integral operational amplifiers. The circuit elements, having more four terminals, are multiwinding transformers, various micro-modules are solid-state components of electron schemes and so on. Three-pole, four pole elements and so on are named the *multipole elements* (multi- terminal networks).

We distinguish active and passive elements of a circuit. The power sources are *active elements*. We also consider electron lamps, transistors, operational amplifiers as active elements, which are capable to strengthen an electrical signal. We consider passive elements, in which energy disperses or accumulates (resistors, inductive coils, capacitors, transformers).

The circuit elements can be described by algebraic or differential equations connecting currents and voltages across the terminals of these elements. The coefficients connecting voltages and currents and their derivatives are named *parameters* of the element.

If the circuit element is characterized by linear algebraic or linear differential equations, it is named *linear*. The parameters of a linear element can be constant or change in time (the *parametric element*). If the circuit element is recorded by non-linear algebraic or non-linear differential equations, it is named *non-linear*. The parameters of a *non-linear element* are non-constant, they depend on the value of the current and of the voltage. The parameters of a non-linear parametric element depend on time too.

In many cases the parameters of an element are considered *lumped* (the elements with lumped parameters), in this case the voltages and the currents of the element are not functions of space coordinates determining the geometric sizes of the element.

The parameters of the element can be also distributed (the elements with *distributed parameters*), such an element is recorded by the equations, in which the voltages and the currents depend on space coordinates.

The elements of an electrical circuit can satisfy or not to satisfy the *principle of reciprocity*. In simple words this principle is read so: the response of the circuit on segment 1 from the disturbance on segment 2 is equal to the response on segment 2 of the same disturbance on segment 1. According to it we distinguish *mutual* and *non-mutual* elements. The examples of mutual elements are resistors, inductive coils, capacitors, transformers; but electron lamps, transistors and others are not non-mutual elements.

The circuits containing only *mutual elements*, are named *mutual* (circuits consisting of resistors, capacitors, inductive coils transformers and power sources). If in a circuit there are non-mutual elements, the circuit is named *non-mutual* (circuits with electron lamps, transistors, operational amplifiers).

The circuits containing only linear elements are named *linear*. Their main property is that we can use the *principle of superposition* (the response of a circuit at operating several disturbances is equal to the sum of responses stipulated by each disturbance separately). If the circuit has even one non-linear element, it is named *non-linear*.

1.2. The scheme of an electrical circuit

For simplifying the mathematical record the real elements of a circuit are idealized. According to its mathematical model the idealized element of the circuit is replaced by a schematic element. Any real element of the circuit can be represented by one or a combination of schematic elements, connected by a certain way. Such a combination of elements (in the specific case when there is one element) is named the *replacing scheme* or the *equivalent scheme* of an electrical circuit element under condition of coinciding the equations describing this scheme and the circuit element.

The conventional geometric image corresponds to each schematic element. Then the way of connecting the elements of a real circuit can be represented by the appropriate connection of schematic elements. The geometric image of connecting schematic elements, showing the connection of real elements of an electrical circuit and its properties, are named the *scheme of a circuit*.

In the scheme we distinguish *branches* - segments, (in any cross-section of a segment the current has the same value), and *nodes* - points, (in which not less than three branches are connected).

1.3. Two-pole active elements (power sources)

Any two-terminal network can be represented conventionally by the following scheme (Fig. .1.1). The two-terminal network is connected to other elements with the help of terminals 1 and 2. The voltage u between these terminals is measured in Volts (*V*), the current i of the two-terminal network is measured in Amperes (*A*).

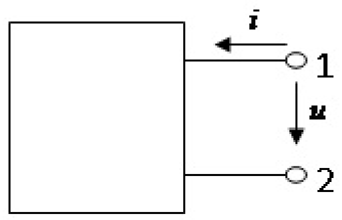

Fig.1.1

The voltage u is identified with the potential drop across the terminals $u = \varphi_1 - \varphi_2$.

The arrows indicate positive directions of the voltage and the current. These directions are chosen arbitrarily. Any equations describing correlation between the voltages and the currents of the element (in the network as a whole) make sense only for the chosen positive directions. The positive direction can be indicated by a double index, for example: u_{12} (thus is $-u_{21} = +u_{12}$).

For the positive directions, shown in Fig. 1.1, the product of the voltage and the current (ui) represents an instantaneous power fed by the two-terminal network: $p = ui$.

The energy, flowing into the element during the time interval $0 - t$, is

$$W = \int_0^t p d\tau = \int_0^t ui d\tau .$$

If for any time t the energy $W \geq 0$, this two-terminal network is a customer of energy and named *passive*.

The *active* two-terminal network generates energy. For such a two-terminal network is $W < 0$.

If in Fig. 1.1 to change the positive direction of the current by the opposite one, the integral $W = \int_0^t ui d\tau$ will determine the generated energy. This case $W > 0$ corresponds to the power source, $W < 0$ corresponds to the load of energy.

The real power sources can be two kinds: the *source of an E.M.F.* (the source of a voltage) and the *source of a current*.

<u>The source of an E.M.F.</u> The source of an E.M.F (the source of a voltage), is characterized by the value of the E.M.F e, equal to the voltage, i.e. the potential drop across the terminals when there is no current through the source of the E.M.F. The E.M.F. is

the work of forces of the source spent on the moving of a unit positive charge inside the source from the terminal with a less potential to the terminal with a larger potential.

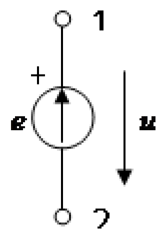

Fig.1.2

If the voltage across the terminals of the source is equal to the E.M.F at any current through the source $u = e$, the source is named *ideal*. The scheme of such a source is given in Fig. 1.2. The arrow inside the circle indicates the positive direction of the forces in the sources (the positive direction of the E.M.F). If e does not depend on time

$$e(t) = E = \text{const},$$

then we have the source of the direct E.M.F. In such a source the beginning of the arrow indicating the positive direction of the E.M.F, corresponds to the negative terminal. The image of the source of the constant E.M.F may be shown as in Fig. 1.3.

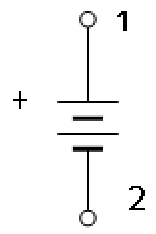

Fig.1.3

The scheme of the real source of a direct E.M.F is shown in Fig. 1.4.

r_{int} is the internal resistance

$$u = E - r_{\text{int}} i \,. \tag{1.1}$$

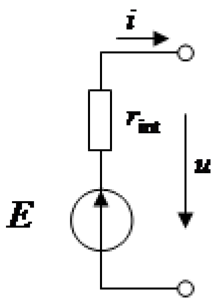

Fig.1.4

The dependence $u = u(i)$ is named the *external characteristic* of the source of an E.M.F (Fig. 1.5).

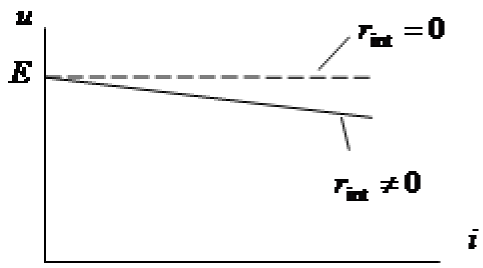

Fig.1.5

The source of a current. The source of a current is characterized by a current i at short-circuit terminals (i.e. in the absence of a voltage across the source terminals). If the current of the source does not depend on the voltage i.e. $i = J$ at any voltage across the terminals, the source of the current is named *ideal*. The scheme is given in Fig. 1.6.

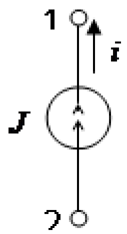

Fig.1.6

________________ The double arrow shows a positive direction of the current in the source of the current. If $J = \text{const}$, we have the source of a direct current. The scheme of the real source of the current is shown in Fig. 1.7.

g_{int} is the internal conductance.

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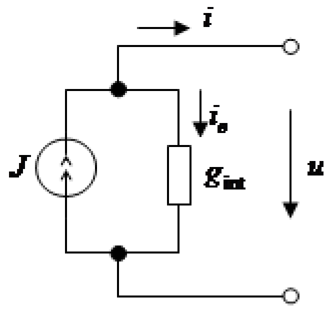


Fig.1.7

$$i = J - g_{\text{int}} u\,. \qquad (1.2)$$

The dependence $i = i(u)$ is named the *external characteristic* of the source of a current (Fig. 1.8.)

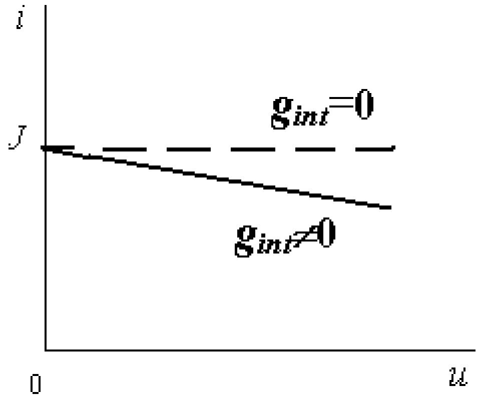


Fig. 1.8

In a number of cases the equivalent circuits of real sources of an alternating E.M.F. and an alternating current can be represented accordingly as follows shown in Fig. 1.4 and Fig. 1.7.

Equivalence of sources.

We can speak about two equivalent circuits of a real power source (Fig. 1.4, 1.7). These schemes are equivalent, if $J = E / r_{\text{int}}$, $g_{\text{int}} = 1/r_{\text{int}}$, i.e. at the same voltage u (the current i) the currents i (the voltage u) of these schemes are equal.

Further, if there are no stipulations, the sources of E.M.F. and currents are understood as ideal sources.

For describing the properties of components of electronic circuits you should introduce the so-called controlled (dependent) sources of E.M.F. and currents, their parameters depend on the voltages or the currents through other segments of the considered electrical circuit. It is possible to introduce four types of such controlled sources:

1) the *source of a voltage controlled by the voltage* (this source may be called the *amplifier of a voltage*) its E.M.F. is proportional to the voltage u_{ab} across two points a and b of the considered circuit (Fig.1.9);

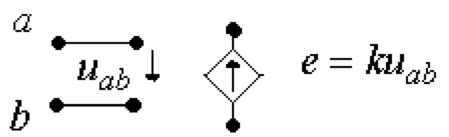


Fig.1.9

2) the *source of a current controlled by the voltage*, its current is proportional to the voltage u_{ab} across that or other segment of the circuit (Fig.1.10);

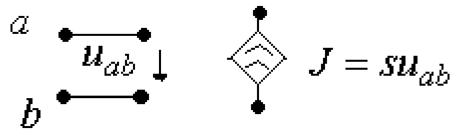


Fig.1.10

3) the *source of a voltage controlled by the current*, its E.M.F. is proportional to the current i_{ab} through some segment of the considered circuit (Fig.1.11);

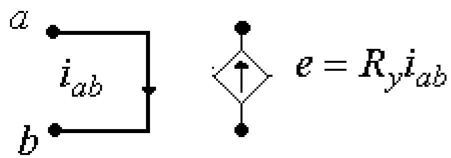


Fig.1.11

4) the *source of a current controlled by the current*, or the *amplifier of a current*, its current J is proportional to the current of the other segment of the circuit i_{ab} (Fig.1.12).

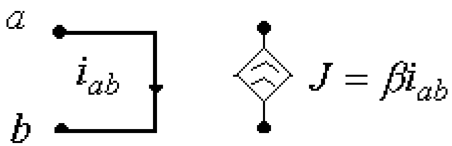


Fig.1.12

1.4. The passive two-terminal network

The main passive two-terminal networks of the scheme are resistive (resistance or conductance), inductive and capacitive elements.

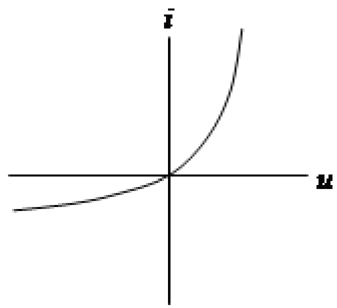


Fig.1.13

The resistive element. A two-terminal network characterized by the dependence $u = u(i)$ or $i = i(u)$, is named the *resistive element*, and the dependence $u = u(i)$ or $i = i(u)$ is named the *Volt-Ampere characteristic* (*VAC*). The non-linear *VAC* is shown in Fig. 1.13 (it is the characteristic for a crystal diode), and in Fig. 1.14 we see a linear *VAC*.

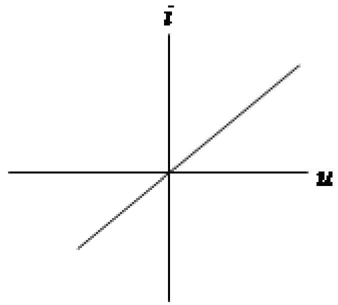


Fig.1.14

The resistance (the conductance) corresponding to Fig. 1.13, depends on the current (the voltage) and it is named the *non-linear resistance* (the conductance). The resistance (the conductance) corresponding to Fig. 1.14, doesn't depend on the current (the voltage), and it is named *linear* (it is constant). The conventional symbols of such a resistance are given in Fig. 1.15. In this case ***Ohm's law*** takes place:

$$u_R = Ri \quad \text{or} \quad i = Gu_R \tag{1.3}$$

where *R* is the resistance; $G = \frac{1}{R}$ is the conductance. A small unit of resistance is microhm (1/10^6 or $10^{-6}\Omega$; large units are kilohm ($10^3\Omega$), symbol kΩ, and megohm (10^6 Ω), symbol MΩ. The unit of conductance is the Siemens, symbol *S*.

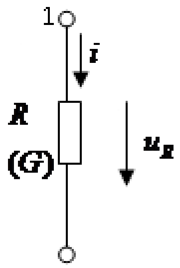


Fig.1.15

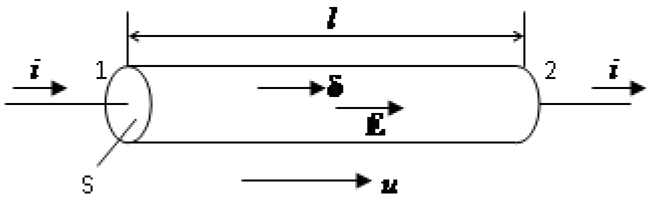


Fig.1.16

The resistance $R > 0$. It is the passive element. The energy, flowing into it, is determined by the formula

$$W = \int_0^t u_R i d\tau = R\int_0^t i^2 d\tau > 0.$$

This energy converts into heat. In this case the power is $p = i^2 R$ (***Joule's law***).

The resistance R as an element of the scheme corresponds to the resistor as an element of the circuit. The example of the resistor can be the conductive cylinder (Fig. 1.16).

The voltage across points 1 and 2 is

$$u = \int_1^2 \mathbf{E} d\mathbf{l}\,.$$

The current is

$$i = \int_1^2 \boldsymbol{\delta} d\mathbf{s}\,.$$

If the current is distributed on the cross-section uniformly ($\boldsymbol{\delta} = const$), and the electric field strength $\mathbf{E}$ is identical by the length, then

$$u = EL\,,\; i = \delta S\,.$$

Due to *Ohm's law in the differential form*

$$\boldsymbol{\delta} = \sigma\mathbf{E}\,;\; \delta = \sigma E\,,$$

where σ is the specific conductivity of the conductor material. $\sigma = \frac{1}{\rho}$, where ρ is a constant for the material of the conductive cylinder. It is called the *resistivity* of that material. The following Table 1.1 gives the value of resistivity for various substances.

Table 1.1

Substance	Resistivity ρ, Ωm (at 20^0 C)
Aluminium	$2{,}82 \cdot 10^{-8}$
Brass	$8 \cdot 10^{-8}$
Constantan	$50 \cdot 10^{-8}$
Copper	$1{,}72 \cdot 10^{-8}$
Iron	$9{,}8 \cdot 10^{-8}$
Manganin	$44 \cdot 10^{-8}$
Mercury	$95{,}77 \cdot 10^{-8}$
Nichrome	$100 \cdot 10^{-8}$
Silver	$1{,}62 \cdot 10^{-8}$
Tungsren	$5{,}5 \cdot 10^{-8}$
Carbon	$(33\text{-}185) \cdot 10^{-8}$

Therefore

$$R = \frac{u}{i} = \frac{El}{\delta S} = \frac{l}{\sigma S}\,.$$

At the alternating current the resistance of the cylinder is increased at the expense of non-uniform distributing the current because of the *skin effect* and the *effect of proximity*.

The inductive element. The two-terminal network characterized by the dependence $\Psi = \Psi(i)$ or $i = i(\Psi)$, is named the *inductive element*, and the dependence $\Psi = \Psi(i)$ is named *Weber-Ampere characteristic* (*WAC*). Ψ is the flux linkage measured in Webers [*Wb*]. In Fig. 1.17 a non-linear *WAC* is given, and in Fig. 1.18 a linear *WAC* is shown.

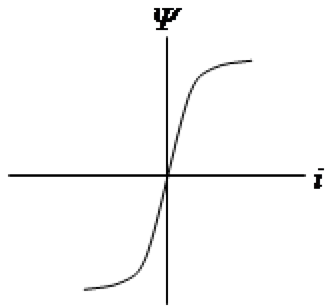


Fig.1.17

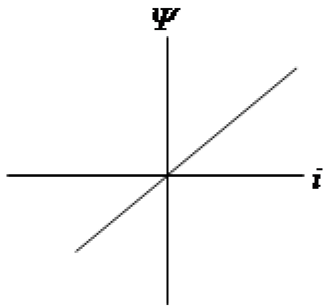


Fig.1.18

The linear inductive element (Fig. 1.18) has the following equation

$$\Psi = Li, \; L = \text{const}.$$

L is the inductance. It has units in Henries [*H*]. The conventional symbol of the linear inductance ($L = \text{const}$) is given in Fig. 1.19.

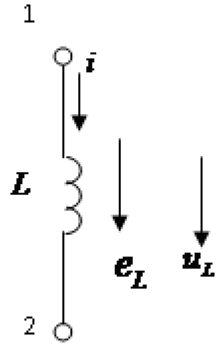


Fig.1.19

The voltage across the terminals of inductance is

$$u_L = \frac{d\Psi}{dt}.$$

In case of linear inductance we can write the following equation

$$u_L = L\frac{di}{dt},\ i = \frac{1}{L}\int_0^t u_L d\tau + \frac{1}{L}u_L(0) = \frac{1}{L}\int u_L d\tau. \qquad (1.4)$$

The inductance L, as a schematic element, corresponds to an inductive coil, which is an element of the circuit.

The inductive coil can be represented as a ring-shaped core, on which the wire is winded uniformly (Fig. 1.20). The current i in the winding creates a *magnetic flux* Φ, closed in the core (the *leakage flux* is neglected). The directions i and Φ are connected by the rule of the right screw. The flux linkage of the coil is $\Psi = w\Phi$, where w is the number of winding turns. The magnetic flux is

$$\Phi = \int_S \mathbf{B}d\mathbf{s},$$

where $\mathbf{B}$ is the *flux density*, $\mathbf{B} = \mu_0 \mu \mathbf{H}$, $\mu_0 = 4\pi \cdot 10^{-7}\ H/m$ is the *permeability* of a free space, μ is the *relative permeability*. By the ***law of a net current***

$$\oint_l \mathbf{H} d\mathbf{l} = iw .$$

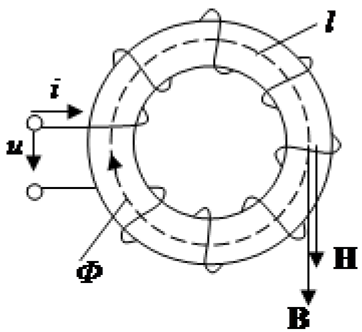


Fig.1.20

If l is essentially larger than cross-sectional sizes of the core, we can consider that the flux Φ is distributed on the cross-section of the core uniformly and therefore

$$\Phi = BS\,,\ \ Hl = iw .$$

Then

$$L = \frac{\Psi}{i} = \frac{wBS}{i} = \frac{w\mu_0 \mu i w S}{li} = \mu_0 \mu \frac{w^2 S}{l} .$$

From (1.4) we can see that u_L is different from zero only at $i \neq \mathrm{const}$ ($\Psi \neq \mathrm{const}$). The changing current creates a changing magnetic flux and by the ***law of an electromagnetic induction*** the E.M.F. (called the E.M.F. of self-induction) is induced in the winding

$$e_L = -\frac{d\Psi}{dt}$$

or at $L = \text{const}$

$$e_L = -L\frac{di}{dt}.$$

The linear inductance ($L = \text{const}$) is the passive element. The energy, flowing into such an element, is determined as follows

$$W = \int_0^t u_L i d\tau = L\int_0^t \frac{di}{d\tau} i d\tau = \frac{1}{2}L\int_0^i d(i^2) = \frac{1}{2}Li^2 > 0$$

under condition $i(0) = 0$. This energy is reserved in the magnetic field of the coil.

The capacitive element. If the two-terminal network is characterized by the dependence $q = q(u)$ or $u = u(q)$, it is named the *capacitive element* (*capacitance*), and the indicated dependencies are called *Coulomb-Volt characteristics*. Here q is an electrical charge, so we have

$$i = \frac{dq}{dt}.$$

In Fig. 1.21 and 1.22 the characteristics for non-linear and linear capacities are shown accordingly.

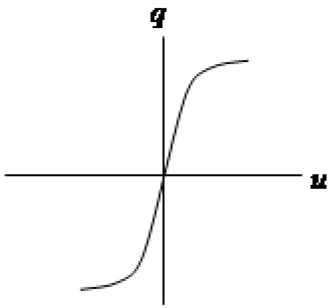


Fig.1.21

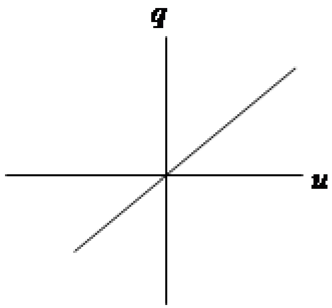


Fig.1.22

For a linear capacitive element the following equation is carried out

$$q = Cu_C,$$

where $C = \frac{q}{u_C}$ is the *capacitance*. It has units in Farads [*F*]. One Farad is the capacitance of an extremely large capacitor. In real circuits, such as radio receivers, the capacitance of the used capacitor is expressed in microfarad (μ*F*). One microfarad is one millionth part of a Farad, that is $1\mu F = 10^{-6} F$. It is also quite usual to express small capacitors, such as those on record players, in picofarad (*pF*). A picofarad is one millionth part a microfarad, that $1\ pF = 10^{-6}\mu F = 10^{-12} F$.

The conventional symbols of the linear capacitance are given in Fig. 1.23.

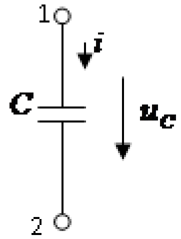


Fig.1.23

The current, running through the capacitance, is $i = \dfrac{dq}{dt}$.

If $C = \text{const}$, we have

$$i = C\frac{du_C}{dt}, \; u_C = \frac{1}{C}\int_o^t i d\tau + u_C(0) = \frac{1}{C}\int i d\tau . \qquad (1.5)$$

The capacitance C, as a schematic element, corresponds to the capacitor as an element of the circuit.

We can represent the capacitor as two parallel conductive plates with the area S divided by a dielectric with the thickness d (Fig. 1.24).

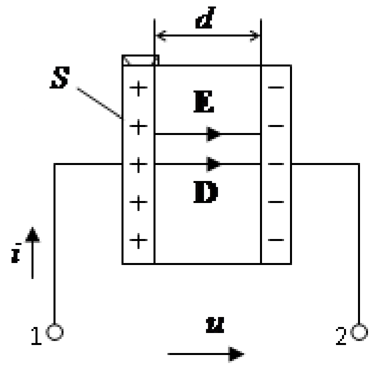


Fig.1.24

At $u=\varphi_1-\varphi_2>0$ there will be a charge $q_+=q$ on the left plate, but on the right plate $q_-=-q$.

$$q_+=\int_S \mathbf{D}d\mathbf{s}\,,\;\; \mathbf{D}=\varepsilon_0\varepsilon\mathbf{E}\,,$$

$\varepsilon_0=1/(4\pi\cdot 9\cdot 10^9)\;\; F\cdot m^{-1}$ is the *permittivity* (or the dielectric constant), ε is the relative permittivity. The following Table 2 gives the value of a relative permittivity for various substances.

If the field in the capacitor is homogeneous the following equation is carried out

$$q=DS=\varepsilon_0\varepsilon ES$$

and the capacitance will be

$$C=\frac{q_+}{\varphi_1-\varphi_2}=\frac{DS}{u}=\frac{\varepsilon_0\varepsilon ES}{Ed}=\varepsilon_0\varepsilon\frac{S}{d}\,.$$

From (1.5) we can see that the current, flowing through the capacitance, is different from zero only at $u_C \neq \text{const}$. The change of the voltage across the plates causes changing the value of a charge on the plates.

The linear capacitance ($C = \text{const}$) represents a passive element. The energy, flowing into it, is

$$W = \int_0^t u_C i d\tau = C\int_0^t u_C \frac{du_C}{d\tau} d\tau = \frac{C}{2}\int_0^{u_C} d(u_C^2) = \frac{Cu_C^2}{2} > 0.$$

At $u_C(0) = 0$. In this case the energy is reserved in the electrical field of the capacitor.

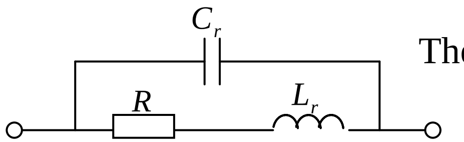
The process of accumulating energy, both in the magnetic field and in the electrical field, is converted. The accumulated energy can be given to other elements (for example, at the discharge of the capacitor through a resistor).

The equivalent circuits of resistors, inductive coils and capacitors.

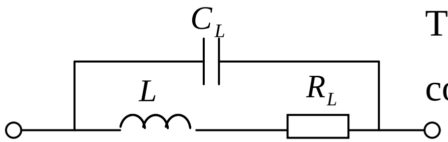
The schematic elements are the resistance R, the inductance L, the capacitance C expressing main properties and parameters accordingly to resistors, inductive coils and capacitors characterizing physical processes of an irreversible dissipating of energy and a reversible accumulating of energy, connected with magnetic and electrical fields. With the help of schematic elements R, L and C we can make equivalent circuits of resistors, inductive coils and capacitors taking even minor processes into account.

Table 1, 2

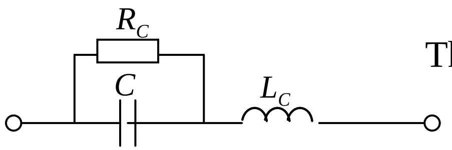
Substance	Relative permittivity
Glass	5-10
Mica	6
Ebonite	2,8
Ice	94
Paraffin wax	2
Paraffined paper	2
Methyl alcohol	32
Water	81
Air (normal pressure)	1,0005



The equivalent circuit of the resistor



The equivalent circuit of the inductive coil



The equivalent circuit of the capacitor

1.5. Main equations of circuits with lumped parameters

These equations follow from the known physical laws: the principle of continuity of a net current and the law of electromagnetic induction.

If some node of the scheme is embraced by a closed surface S (Fig. 1.25), because of the principle of continuity of a net current

$$\oint_S \boldsymbol{\delta}_{net} d\mathbf{s} = 0, \qquad (1.6)$$

where $\boldsymbol{\delta}_{net}$ is the density of a net current, that is the sum of a *conduction current* and a *bias current*. In the scheme with lumped parameters the bias current exists only between the plates of capacitors, therefore in (1.6) the density of a net current is equal to the density of a conduction current: $\boldsymbol{\delta}_{net} = \boldsymbol{\delta}$. $\boldsymbol{\delta}$ is not equal to zero in those points of the surface S, which coincides with the cross-section of the conductors. Taking it into account from (1.6) we shall receive

$$\sum_k i_k = 0 \tag{1.7}$$

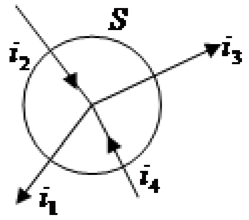


Fig.1.25

The equation (1.7) is named ***Kirchhoff's first law***: the algebraic sum of currents of the branches, connected in the node, is equal to zero at any moment of time. In this case the currents are recorded with the positive sign if they are directed from the node or from the closed surface

$$i_1 - i_2 + i_3 - i_4 = 0 .$$

Kirchhoff's first law is fair for the closed surface too, embracing some nodes. In this case in (1.7) the currents of the branches crossed by the surface are summed up. This equation (1.7) can be recorded as follows

$$\sum_k i_k = \sum_k J_k \tag{1.8}$$

where $\sum_k J_k$ is the algebraic sum of the currents of current sources; $\sum_k i_k$ is the algebraic sum of the currents of other branches (elements). In (1.8) the positive sign is given to J_k, directed to the node; the positive sign is given to i_k, directed from the node.

Due to the *law of electromagnetic induction* for any closed loop l we have

$$\oint_l \mathbf{E}d\mathbf{l} = -\frac{d\Phi}{dt}. \qquad (1.9)$$

The directions of the integrating and the flux are coordinated by the rule of the right screw.

Let's take a closed loop l on the scheme of the circuit, so that it should pass outside the sources and the inductances. As in the circuit with lumped parameters the magnetic field is lumped in inductance, then the indicated loop l (1.9) gives

$$\oint \mathbf{E}d\mathbf{l} = 0 \qquad (1.10)$$

I.e. the field of the vector $\mathbf{E}$ is potential and the voltage across any two points of the loop coincides with the potential drop. From (1.10) it follows

$$\sum_k u_k = 0 \qquad (1.11)$$

This equation is named ***Kirchhoff's second law***: the algebraic sum of voltage branches (elements) of the loop is equal to zero at any moment of time. With a positive sign you take the voltage, its positive direction coincides with the direction of the path-tracing of the loop.

Let's take a closed loop 1a23b41 (Fig. 1.26). In this case the equation (1.11) is

$$-u_1 + u_2 - u_3 + u_4 = 0,$$

$$u_1 = \varphi_2 - \varphi_1, \quad u_2 = \varphi_2 - \varphi_3, \quad u_3 = \varphi_4 - \varphi_3, \quad u_4 = \varphi_4 - \varphi_1.$$

If in (1.11) the voltages of the sources are transferred to the right part and replace them by E.M.F. we shall obtain the following

$$\sum_k u_k = \sum_k e_k, \qquad (1.12)$$

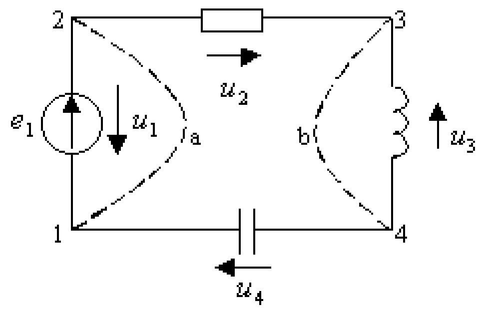


Fig.1.26

I.e. the algebraic sum of the voltages across the passive elements $\sum_k u_k$ is equal to the algebraic sum of the E.M.F. in the loop $\sum_k e_k$. In (1.12) the voltages and the E.M.F. which have the directions, coinciding with the direction of the path-tracing of the loop, are recorded with the positive sign.

The example.

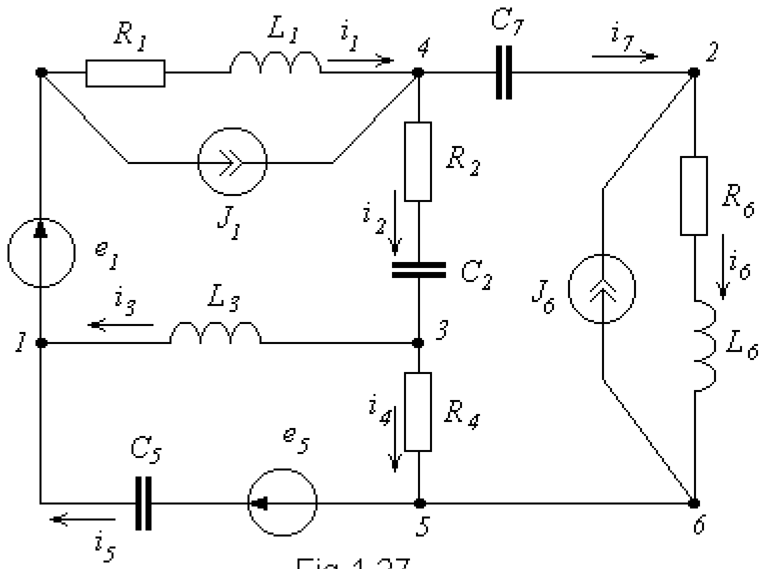


Fig.1.27

Record Kirchhoff's equations for the scheme (Fig. 1.27).

For node 1 $i_1 - i_3 - i_5 = -J_1$. For node 2 $i_6 - i_7 = J_6$. For node 3 $i_3 + i_4 - i_2 = 0$. For node 4 $i_2 + i_7 - i_1 = J_1$.

For the loop $R_1 + L_1 + R_2 + C_2 + L_3 + e_1$ at the path-tracing is clockwise:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + R_2 i_2 + \frac{1}{C_2}\int_0^t i_2 dt + L_3 \frac{di_3}{dt} = e_1.$$

For the loop $L_3 + R_4 + C_5 + e_5$ at the path-tracing is clockwise:

$$-L_3 \frac{di_3}{dt} + R_4 i_4 + \frac{1}{C_5}\int_0^t i_5 dt = e_5.$$

For the loop $R_1 + L_1 + C_7 + R_6 + L_6 + C_5 + e_1 + e_5$ at the path-tracing is clockwise:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_7}\int_0^t i_7 dt + R_6 i_6 + L_6 \frac{di_6}{dt} + \frac{1}{C_5}\int_0^t i_5 dt = e_1 + e_5.$$

1.6. The graph of an electrical circuit

The electrical circuit is characterized by a set of elements and by the way of their connection. Let's consider equations of an electrical circuit determining the way of connecting elements.

Let the circuit consist of two-terminal networks. In the simplest case these elements can be connected in series or in parallel.

At a series connection any two adjacent elements have a common terminal (Fig. 1.28), the currents in all the elements are identical. The voltage across the terminals of the whole connection is equal to the sum of the voltages of separate elements

$$u = u_1 + u_2 + \ldots + u_n = \sum_k^n u_k \ . \tag{1.13}$$

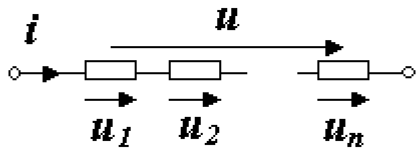


Fig 1.28

At a parallel connection all the elements are connected to the same couple of nodes, so that the voltages across all the elements should be identical (Fig. 1.29). The current in the non-branched part of the circuit is equal to the sum of the currents of all the elements

$$i = i_1 + i_2 + ... + i_n = \sum_k^n i_k \,. \tag{1.14}$$

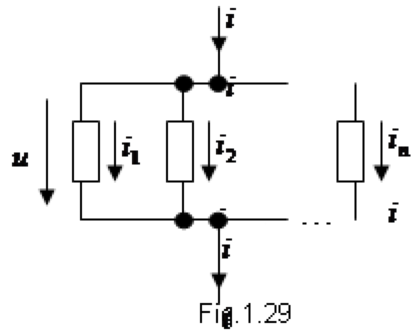


Fig.1.29

The equations (1.13) and (1.14) are fair for connections of any elements (resistive, inductive etc.), and they are determined only by the way of connecting elements, or, they say, by geometry (topology) of connections.

The *topological (geometric) properties* of an electrical circuit do not depend on the type and properties of elements, which the circuit consists of. Therefore each branch can be represented as a line segment. Fig. 1.31 represents the *graph* of an electrical circuit shown in Fig. 1.30.

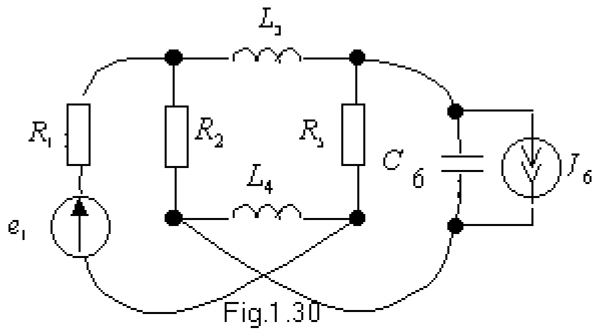


Fig.1.30

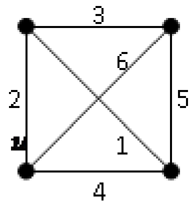


Fig.1.31

On the graph the sources of the E.M.F. and the current are not shown. In this case a branch with the source of an E.M.F. is kept, a branch with the source of a current is not shown. The line segments are named the *branches* of the graph, the end points of a branch are named the *nodes of the graph*.

The branches of the graph may have a certain orientation shown by the arrows. The graphs, which have all oriented branches, are named *oriented*. Otherwise they are named *non-oriented*.

The *subgraph* is named a part of the graph (one branch, one node, any set of branches and nodes of the graph). In the theory of circuits subgraphs are distinguished: a track, a loop, a tree, leaves, cutsets.

The *track* is a certain sequence of branches, in which every two adjacent branches have a common node, and any branch and any node meet once in this track. In Fig. 1.31 we have tracks: 3-4-5; 2-1-5 etc.

The *loop* is a closed track, in which one of the nodes is both the starting node and the final node of the track. In Fig. 1.31 we have loops: 2-3-6, 2-3-5-4 etc.

If there is a track between any couple of nodes of the graph (scheme) such a graph (scheme) is named *linking*.

The *tree* of a linking graph (scheme) is named the *linking subgraph* (*subcircuit*) containing all the nodes of the graph (scheme), but there are no loops at all. The trees shown in Fig. 1.32 are corresponding to the graph in Fig. 1.31.

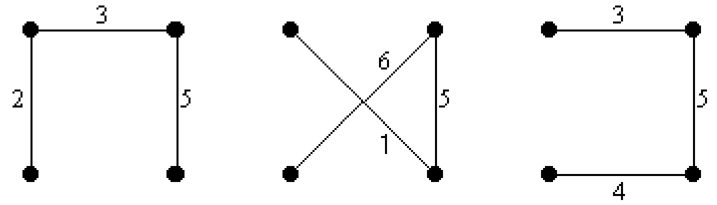


Fig. 1.32

The branches of the graph (scheme), adding the tree to the initial graph, are named the *branches of leaves* (additions of the tree). The branches of leaves of trees shown in Fig. 1.33, are corresponding to the trees in Fig. 1.32.

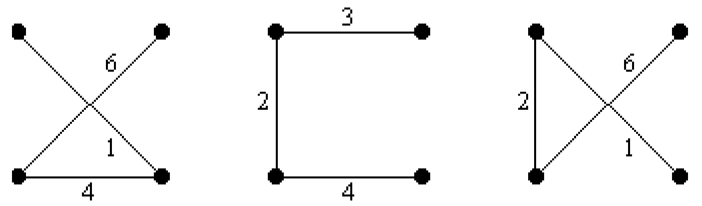


Fig.1.33

If a graph (scheme) contains "p" branches and "q" nodes, the number of branches of any tree $g = q - 1$. The number of branches of leaves of the graph $k = p - (q - 1)$.

The *cutset* of a graph (scheme) is named a set of branches, when we delete them, the graph (scheme) is divided into two isolated subgraphs (subcircuit), one of them can be an isolated node in the special case. The cutset can be shown as a track of the closed surface, which crosses the corresponding branches. Such surfaces (S_1; S_2) are shown in Fig. 1.34.

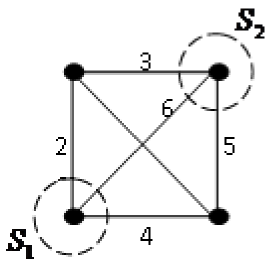


Fig. 1.34

The *main loop* is named a loop consisting of branches of a tree and one branch of leaves. The number of main loops is $p-(q-1)$.

The *main cutset* is named a cutset consisting of branches of leaves and one branch of the tree. Each branch of a tree allows to form one main cutset. The number of main cutsets is $q-1$.

In Fig. 1.35 the tree is shown by thick lines. The main loops are: 2-3-6; 3-5-1; 2-3-5-4. The main cutsets are shown by the surfaces S_1, S_2 and S_3.

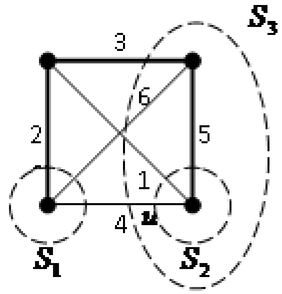


Fig.1.35

The problem of the number of independent equations, which can be made by Kirchhoff's laws may be solved using the concept of a graph tree. One can show, that: the equations, recorded for main cutsets by Kirchhoff's first law, are independent. The equations, recorded for main loops by Kirchhoff's second law, are also independent.

$q-1$ independent equations are made by Kirchhoff's first law.

$p-(q-1)$ independent equations are made by Kirchhoff's second law.

1.7. Topological matrixes of a graph

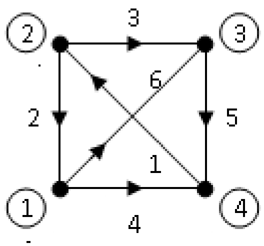


Fig. 1.36

Let's consider an orientated graph (Fig. 1.36), corresponding to the scheme (Fig. 1.30). .Let's make Kirchhoff 's independent equations for nodes 1, 2, 3.

$$\begin{cases} \quad -i_2 + i_4 + i_6 = 0 \\ -i_1 + i_2 + i_3 = 0 \\ \quad -i_3 + i_5 - i_6 = 0 \end{cases}$$

These equations can be recorded in the matrix form:

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the matrix of currents

the topological matrix

The elements of the topological matrix are equal +1, -1, 0. The values of these elements are determined only by the graph (scheme) structure.

Due to the type of Kirchhoff's equations we can distinguish three topological matrixes: the matrix of connections (a node matrix) $\mathbf{A}$, the matrix of cutsets $\mathbf{Q}$ and the matrix of loops $\mathbf{B}$.

The *matrix of connections* (a node matrix) $\mathbf{A}$ is a table of the coefficients of equations made by Kirchhoff's first law for nodes.

The rows of the matrix correspond to the nodes (the number of rows is equal to the number of independent nodes, i.e. $q-1$), the number of columns is equal to the number of branches. If we record $\mathbf{A} = \left[\, a_{ij} \right]$, $\mathbf{A}$ is the node matrix (i is the row number, j is the column number).

The rule of designing the matrix $\mathbf{A}$: $a_{ij} = 1$, if the branch j is connected with the node i and directed from the node i; $a_{ij} = -1$, if the branch j is connected with the node i and is directed to the node i, $a_{ij} = 0$, if the branch j is not connected with the node i.

If we mark

$$\mathbf{i}^{(\mathbf{p})} = \begin{bmatrix} i_1 \\ \cdot \\ \cdot \\ i_p \end{bmatrix},$$

the matrix form of equations for nodes by Kirchhoff's first law will be

$$\mathbf{A}\mathbf{i}^{(\mathbf{p})} = \mathbf{0}\,. \tag{1.15}$$

With the help of the matrix $\mathbf{A}$ we can record the voltages of branches (u_1, u_2, …, u_p) through the potentials of nodes (φ_1, φ_2, …, φ_q)

$$\mathbf{u}^{(\mathbf{p})} = \mathbf{A}^{\mathbf{T}}\varphi\,, \tag{1.16}$$

where $\mathbf{u}^{(\mathbf{p})} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_p \end{bmatrix}, \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \varphi_q \end{bmatrix}$, $\mathbf{A}^{\mathbf{T}}$ is the transposed node matrix.

The matrix of cutsets $\mathbf{Q}$ is a table of the coefficients of the equations made by Kirchhoff's first law for cutsets. The rows of the matrix correspond to the cutsets, the columns correspond to the branches

$$\mathbf{Q} = [q_{ij}]\,.$$

$q_{ij} = 1$, if the branch j is in the cutset i and directed accordingly with the direction of the cutset (the current of a tree branch j is directed outside the closed surface). $q_{ij} = -1$, if the branch j is in the cutset i and directed in the opposite direction of the cutset; $q_{ij} = 0$, if the branch j is not in the cutset i.

The number of rows in the matrix $\mathbf{Q}$ is equal to the number of independent nodes $q-1$.

If the matrix $\mathbf{Q}$ is made for main cutsets, it is named the *matrix of main cutsets*. In this case the positive direction of the cutset is the direction of the tree branch of the given cutset (Fig. 1.37).

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix},$$

where S_1, S_2, S_3 are the independent cutsets.

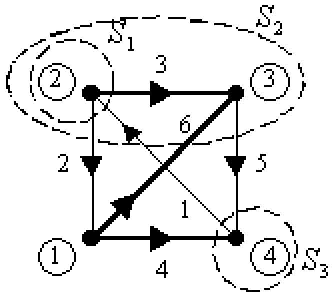


Fig.1.37

Kirchhoff's law for the cutsets in the matrix form is

$$\mathbf{Qi}^{(\mathbf{p})} = \mathbf{0}. \tag{1.17}$$

If the matrix of voltages of tree branches is marked by $\mathbf{u}^{(Д)}$, we obtain

$$\mathbf{u}^{(\mathbf{p})} = \mathbf{Q}^{\mathbf{T}}\mathbf{u}^{(Д)}. \tag{1.18}$$

The *matrix of loops* $\mathbf{B}$ is a table of the coefficients of the equations made by Kirchhoff's second law. The rows of the matrix correspond to the loops, the columns correspond to the branches.

$$\mathbf{B} = [b_{\mathrm{ij}}].$$

$b_{ij} = 1$, if the branch j is in the loop i and the direction of the branch coincides with the direction of the path-tracing of the loop; $b_{ij} = -1$, if the branch j is in the loop i and its direction is against the direction of the path-tracing of the loop; $b_{ij} = 0$, if the branch j is not in the loop i.

The matrix $\mathbf{B}$ written for main loops is named the *matrix of main loops*. The direction of the *path-tracing* of the loop coincides with the direction of a branch of leaves of this loop.

The number of rows of the matrix $\mathbf{B}$ is equal to the number of independent loops $p-(q-1)$.

Kirchhoff's second law in the matrix form is

$$\mathbf{Bu^{(p)}} = \mathbf{0}, \tag{1.19}$$

$$\mathbf{i^{(p)}} = \mathbf{B^T i^{(k)}}, \tag{1.20}$$

where $\mathbf{i^{(k)}}$ is the matrix-column of loop currents.

The matrixes $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{B}$ allow to express the topology of schemes by the algebraic language, it is important when we analyse complex circuits with the help of computer.

Ratios between topological matrixes.

If the matrixes $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{B}$ are made for the same graph (scheme), we can write

$$\mathbf{AB^T} = \mathbf{0}, \tag{1.21}$$

$$\mathbf{QB^T} = \mathbf{0}. \tag{1.22}$$

1.8. The dual circuits

The dual graphs. Two graphs are named the *dual graphs*, if the node matrix $\mathbf{A}_1$ of one of them is equal to the loop matrix $\mathbf{B}_2$ of the other graph (and on the contrary):

$$\mathbf{A}_1 = \mathbf{B}_2, \qquad (1.23)$$

$$\mathbf{B}_1 = \mathbf{A}_2. \qquad (1.24)$$

The dual elements of the scheme. Two-terminal networks of the scheme are named *dual*, if the dependence $u = u(i)$ of one element coincides with the dependence $i = i(u)$ of the other one and on the contrary.

The source of an E.M.F. $e(t)$ and the source of a current $J(t)$ are dual if $e(t) = J(t)$. We must understand it as the *numerical equality* (the E.M.F. - in Volts and the current - in Amperes).

For the linear resistance R the conductance G $(G = R)$ will be dual and on the contrary. In fact, when $G = R$ the equation $u = Ri$ coincides with the equation $i = Gu$. We must understand this coincidence so: if in the first equation the voltage (the current) is replaced by the current (the voltage), we receive the second equation. We must understand the equality $G = R$ as a numerical equality of conductivities (in Simens) and the resistance (in Ohms).

Non-linear dependencies $u(i)$ and $i(u)$ must coincide in the dual non-linear resistive two-terminal networks.

The linear inductive and capacitive two-terminal networks are characterized by the equations

$$u = L\frac{di}{dt}, \qquad i = C\frac{du}{dt},$$

whence we see, that L and C are dual elements when $L = C$.

We must understand the equality $L = C$ as a numerical equality of inductance (in Henries) and capacitance (in Farads). In case of non-linear inductive and capacitive two-terminal networks the duality means the coincidence of the non-linear characteristics $\Psi(i)$ and $q(u)$.

Dual circuits. Two schemes of electrical circuits containing two-terminal networks are named *dual*, if they have dual graphs and each element of one scheme corresponds to a dual element of the other one.